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Analysis of anomalous diffusion in the Kuramoto-Sivashinsky equation

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In the present paper, based upon the reasonable assumption that the particle displacement $\Delta r(t)$ has a Gaussian distribution, anomalous diffusion in the Kuramoto-Sivashinsky equation is analyzed. There is good agreement between the predictions of this paper and the simulation results of Bohr and Pikovsky [Phys. Rev. Lett. **70**, 2892 (1993)].

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Motion of advected particles in turbulent fluids is of both fundamental and practical importance. Because of the complexity of three-dimensional turbulence, it is useful to investigate a simpler model system that manifests some properties of the actual turbulence. A particular example is the Kuramoto-Sivashinsky (KS) equation

$$\frac{\partial u}{\partial t}+u\nabla u=-\nabla^2 u-\nabla^4 u\ , \tag{1}$$

which has been derived in a variety of contexts from chemical turbulence to flame-front propagation. Investigating diffusion in the KS equation is helpful for understanding diffusion in turbulent flows. The motion of a single marked particle satisfies

$$\frac{dr(t)}{dt}=u(r(t),t)\ , \tag{2}$$

in which $u(x,t)$ is a velocity field and $r(t)$ is the trajectory of the marked particle immersed in the fluid. Bohr and Pikovsky [1] have recently studied Eqs. (1) and (2) using numerical and analytical techniques. Their numerical simulations show that particle displacement $\Delta r(t)\sim t^{\eta}$ where $\frac{4}{3}<\eta<\frac{3}{2}$, and the best fit gives $\eta=1.38$. In order to reveal the results of the numerical simulations, they used the mean field approximation to analyze anomalous diffusion in the KS equation.

It is well known that the large-scale properties of the one-dimensional Kuramoto-Sivashinsky equation can be described by the Burgers equation with white noise [2–5]

$$\frac{\partial u}{\partial t}+u\nabla u=\nu\nabla^2 u+\eta(x,t)\ , \tag{3}$$

where the white noise $\eta(x,t)$ is specified by the two-point correlation

$$\langle \eta(x,t)\eta(x',t')\rangle=-\Gamma\nabla^2\delta(x-x')\delta(t-t')\ . \tag{4}$$

For short times and small systems, the statistical properties of (3) can be described by the linear version [1,6]

$$\frac{\partial u}{\partial t}=\nu\nabla^2 u+\eta(x,t)\ . \tag{5}$$

The two-point correlation function for Eq. (5) is

$$\langle u(x+r,t+\tau)u(x,t)\rangle \longrightarrow \frac{\Gamma}{4(\pi\nu^3\tau)^{1/2}}e^{-r^2/4\nu\tau}\ , \tag{6}$$

and the two-point correlation function for Eq. (3) is [7]

$$\langle u(x+r,t+\tau)u(x,t)\rangle \longrightarrow \tau^{-1/z}H\left[\frac{r^z}{\tau}\right]\ , \tag{7}$$

where

$$\lim_{\xi\to\infty}H(\xi)=0\ ,\quad \lim_{\xi\to 0}H(\xi)=\text{const}\ . \tag{8}$$

Using the above relations, Bohr and Pikovsky analyzed anomalous diffusion in Eqs. (5) and (3) by mean field approximation. The diffusion in Eq. (5) is anomalous with exponent $\eta=\frac{3}{2}$ up to the crossover time t_0, and then tends to the behavior $\langle[\Delta r(t)]^2\rangle\sim t\ln t$. The diffusion in Eq. (3) is anomalous with exponent $\eta=\frac{4}{3}$. However, the validity of the mean field approximation is not clear. In this paper, based upon a reasonable assumption, the problems of anomalous diffusion in the Kuramoto-Sivashinsky equation are studied.

Batchelor [8] has pointed out that the trajectory $r(t)$ of a single marked particle immersed in a turbulent fluid has a Gaussian distribution for long diffusion times. Also, many experimental data show that it is valid for all diffusion times [9]. Thus, in this paper, we can assume that the particle displacement $\Delta r(t)$ has the Gaussian distribution

$$P[\Delta r(t)]=\frac{1}{\sqrt{2\pi}\sigma(t)}\exp\left\{-\frac{[\Delta r(t)]^2}{2[\sigma(t)]^2}\right\}\ , \tag{9}$$

in which

$$\sigma(t)=\{\langle[\Delta r(t)]^2\rangle\}^{1/2}\ .$$

The diffusion can be expressed by the two-point correlation function [10]

$$\langle[\Delta r(t)]^2\rangle \longrightarrow 2t\int_0^t\langle u(r(\tau),\tau)u(r(0),0)\rangle d\tau\ . \tag{10}$$

First, we analyze the diffusion in the linear Eq. (5). From (6) and (9), we have

$$\begin{aligned}\langle u(r(\tau),\tau)u(r(0),0)\rangle &=\int_{-\infty}^{+\infty}\frac{\Gamma}{4(\pi\nu^3\tau)^{1/2}}\exp\left\{-\frac{[\Delta r(\tau)]^2}{4\nu\tau}\right\}\\ &\quad\times P[\Delta r(\tau)]d[\Delta r(\tau)]\\ &=\frac{\Gamma}{2\sqrt{2\pi}\nu}\frac{1}{\sqrt{\sigma^2+2\nu\tau}}\ .\end{aligned} \tag{11}$$

Substituting (11) into (10) yields

$$\sigma^2=\frac{\Gamma t}{\sqrt{2\pi\nu}}\int_0^t \frac{1}{\sqrt{\sigma^2+2\nu\tau}}d\tau \ . \tag{12}$$

From (12), we obtain

$$\frac{d}{dt}\left[\frac{\sigma^2}{t}\right]=\frac{\Gamma}{\sqrt{2\pi\nu}}\frac{1}{\sqrt{\sigma^2+2\nu t}} \ . \tag{13}$$

In the limit $t\rightarrow\infty$, it is easy to find $\sigma^2=\langle[\Delta r(t)]^2\rangle\sim t^{4/3}$. The diffusion in the linear Eq. (5) is anomalous with the exponent $\eta=\frac{4}{3}$. By the mean field approximation, Bohr and Pikovsky have predicted that the diffusion in the linear Eq. (5) is anomalous with the exponent $\eta=\frac{3}{2}$, up to the time $t_0=16\pi\nu^6/\Gamma^2$, and then tends to the behavior $\sigma^2\sim t\ln t$. Thus the linear equation result in this paper is different from that of Bohr and Pikovsky by the mean field approximation.

Similarly, for the nonlinear Eq. (3), we obtain

$$\frac{d}{dt}\left[\frac{\sigma^2}{t}\right]=2t^{-1/z}H_1\left[\frac{\sigma^2}{t}\right] , \tag{14}$$

in which

$$H_1\left[\frac{\sigma^z}{t}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}H\left[\frac{\sigma^z}{t}r^z\right]\exp\left[-\frac{r^2}{2}\right]dr \ . \tag{15}$$

Utilizing Eqs. (8), (14), and (15), for $z=\frac{3}{2}$, we have $\sigma^2=\langle[\Delta r(t)]^2\rangle\sim t^{4/3}$. This result is the same as that reached by Bohr and Pikovsky using the mean field approximation. The numerical simulations of Bohr and Pikovsky show the high moments $\langle[\Delta r(t)]^{2p}\rangle\sim t^{\eta p}$ up to $p=8$, and there is no sign of multifractality. If the assumption $\Delta r(t)$ has the Gaussian distribution, it is easy to obtain $\langle[\Delta r(t)]^{2p}\rangle\sim t^{\eta p}$.

In conclusion, we use an alternative technique to study anomalous diffusion in the KS equations. We found that the diffusion in either the linear equation or the nonlinear equation is anomalous with exponent $\eta=\frac{4}{3}$. There is good agreement between the prediction of this paper for the nonlinear case and the numerical simulations of Bohr and Pikovsky [1].

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